

From Free Fields to AdS – Thermal Case

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Abstract

We analyze the reorganization of free field theory correlators to closed string amplitudes investigated in [1, 2, 3, 4] in the case of Euclidean thermal field theory and study how the dual bulk geometry is encoded on them. The expectation value of Polyakov loop, which is an order parameter for confinement-deconfinement transition, is directly reflected on the dual bulk geometry. The dual geometry of confined phase is found to be AdS space periodically identified in Euclidean time direction. The gluing of Schwinger parameters, which is a key step for the reorganization of field theory correlators, works in the same way as in the non-thermal case. In deconfined phase the gluing is made possible only by taking the dual geometry correctly. The dual geometry for deconfined phase does not have a non-contractible circle in the Euclidean time direction.

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1 Introduction

The large N gauge theory-closed string duality conjecture has been providing us a lot of deep insights into both gauge theories and gravity. Yet despite of the recent developments in some specific examples [5, 6, 7, 8, 9, 10, 11], we still don't know how to directly translate the descriptions in one side to the other.

Maldacena's conjecture [12, 13, 14] suggests that weakly coupled large N conformal field theories are dual to closed string theories on highly curved AdS space. String theory on AdS space is not yet developed sufficiently to test the Maldacena's conjecture in a complete precision, but the tractability of field theory in weak coupling makes us hope that it will be possible to construct a string theory directly from a field theory in this limit.¹

¹There appeared a lot of literature on this weak/free field theory-string theory correspondence recently. See [15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30] for other approaches in this direction. There is also a lot of literature on the connection between weakly coupled $\mathcal{N} = 4$ Yang-Mills theory and integral spin chain since the work of [31]. For a recent approach to the reorganization of field theory correlators to string worldsheets in light-cone gauge, see [32, 33].

In a series of papers [1, 2, 3, 4], taking free field theory as a starting point Gopakumar presented a refined argument of 't Hooft's reorganization [34] of large N field theory correlators to closed string amplitudes, and made more precise study of the gauge theory-closed string duality possible. The key step was gluing of propagators in each edge of a given Feynman diagram to make up a so called skeleton graph. Each edge is assigned one effective Schwinger parameter after the gluing. Then a mathematical result [35, 36, 37] tells us that the space of Schwinger parameters gives cell decomposition of $\mathcal{M}_{g,n} \times R_+^n$, where $\mathcal{M}_{g,n}$ is a moduli space of genus g Riemann surface with n punctures. The appearance of $\mathcal{M}_{g,n}$ is a strong support that the large N field theory correlators do organize themselves to closed string amplitudes. The resulting closed string is propagating in AdS geometry, as expected from the conformal symmetry [12].

It is interesting to test the generality of the method to see how the information of string theory, for example difference of backgrounds, is encoded on field theory correlators. In particular, recently thermodynamics of weakly coupled Yang-Mills theories have been studied extensively [15, 18, 38, 39, 40, 41, 42, 43, 44, 45] and it is interesting to ask what are the dual geometries in these cases.² Although there are qualitative agreements between phase transitions in Yang-Mills theories and expected phase transitions in corresponding bulk geometries, since weakly coupled gauge theories are dual to highly curved space-times with curvature scales being the order of string scale one cannot rigorously analyze the geometry without string theory. For this reason the reorganization of field theory correlators into closed string amplitudes is a promising approach to study these highly stringy geometries.

In this article, we analyze the reorganization of free field theory correlators in Euclidean thermal field theory. Two typical phases of gauge field theories at finite temperature are confined phase and deconfined phase. The order parameter of the phase transition is Polyakov loop. We find that the expectation value of Polyakov loop is crucial for reconstructing the bulk geometry. In section 2 we study confined phase. The dual geometry of confined phase is found to be an AdS space periodically identified in Euclidean time direction (thermal AdS), as expected. However, this is not just a simple consequence of the periodic identification in field theory side alone. The fact that the expectation value

²We understand that B. Rai has also raised the question of understanding the thermal geometry from the field theory.

of the Polyakov loop is zero in confined phase is crucial for reconstructing the bulk AdS geometry. The gluing of Schwinger parameters, which was a key step for reorganizing field theory correlators to closed string amplitudes [2, 3], works in the same way as in the non-thermal case. In section 3 we turn to deconfined phase. It turns out that the gluing of Schwinger parameters does not work straightforwardly in deconfined phase. We analyze the reason and identified it with a behavior of "string bits" [46, 47], which is reminiscent of that in the Hagedorn transition [48, 49, 50, 51]. We will argue that we can nevertheless glue the field theory correlators if we take the dual geometry correctly and examine the meaning of the gluing in more general context. The dual geometry of deconfined phase is not easily found in general, but in section 4 we present a simple example where we can explicitly find the dual bulk geometry. This is a two dimensional CFT on $S^1 \times R$, where S^1 is the thermal circle. Our general arguments on deconfined phase and its dual geometry in section 3 are concretely realized in this example.

2 Confined Phase vs. Bulk Geometry

In this section we will explain how in confined phase field theory correlators see the dual bulk geometry.³ The result is rather simple: If they see some dual bulk geometry at zero temperature, at finite temperature they just see the same geometry with periodic identification in Euclidean time direction. However, this is not a simple consequence of the periodic identification in field theory side alone. The expectation value of Polyakov loop plays a crucial role. The expectation value of Polyakov loop is an order parameter of confinement-deconfinement transition, and in AdS/CFT correspondence it has a dual description in terms of string worldsheet in the bulk whose end is on the loop [53, 54, 55, 56, 57]. It is interesting to observe how Polyakov loop directly reflects the bulk geometry in our approach.

³Discussions in this section apply to interacting field theories also, regardless of the title of this article. Confinement is usually regarded as a strong coupling phenomenon and it is somewhat counterintuitive to study it in the free field limit which is a main focus of this article. However, it has been shown that if one takes (2.1) below (or $\langle |\mathcal{P}|^2 \rangle = 0$ for gauge theories on a compact spacial manifold) as a criterion for confinement it does occur in some weakly coupled gauge theories on a compact manifold at large N . A good explanation on this account is given in [38], see also the summary [52].

For concreteness, let us study massless scalar field Φ in adjoint representation of gauge group $SU(N)$ on $S^1 \times R^{d-1}$, where S^1 is the thermal circle parameterized by τ with period β . We will work in the gauge where A_0 is constant and diagonal. A criterion for confinement is that the Polyakov loop expectation value vanishes:

$$\langle \mathcal{P} \rangle = 0, \quad \mathcal{P} \equiv \frac{1}{N} \text{Tr} P \exp i \int_0^\beta d\tau A_0 \quad (2.1)$$

where P denotes the path ordering.⁴ This is realized by the following configuration:

$$A_0 = \frac{2\pi}{\beta N} \left(\text{diag}(1, \dots, N) - \frac{N+1}{2} \right). \quad (2.2)$$

In the 't Hooft limit $N \rightarrow \infty$, $g_{YM} \rightarrow 0$ with $\lambda = g_{YM}^2 N$ fixed, the action is generically of order N^2 whereas there are N diagonal components of A_0 , so fluctuations of them around saddle points are suppressed. Whether the configuration (2.2) is realized as the most dominant saddle point depends on the theory (other matter contents etc.). However, once (2.2) is realized the following argument can be applied regardless of the details of the theory. So we assume (2.2) and study its consequence. Let us study perturbative expansion around (2.2).⁵ The quadratic part of the action in the presence of A_0 zero-mode configuration (2.2) is

$$\begin{aligned} & \frac{1}{g_{YM}^2} \int_0^\beta d\tau \int d^{d-1}x D_\mu \Phi_{ab} D^\mu \Phi_{ba} \\ &= \frac{1}{g_{YM}^2 \beta} \sum_{n=-\infty}^\infty \int d^{d-1}p \left(\left(\frac{n + (A_{0aa} - A_{0bb})}{\beta/2\pi} \right)^2 + p^2 \right) \Phi_{ab}\left(\frac{2\pi n}{\beta}, p\right) \Phi_{ba}\left(-\frac{2\pi n}{\beta}, -p\right) \\ &= \frac{1}{g_{YM}^2 \beta} \sum_{n=-\infty}^\infty \int d^{d-1}p \left(\left(\frac{n + \frac{a-b}{N}}{\beta/2\pi} \right)^2 + p^2 \right) \Phi_{ab}\left(\frac{2\pi n}{\beta}, p\right) \Phi_{ba}\left(-\frac{2\pi n}{\beta}, -p\right). \end{aligned} \quad (2.3)$$

Thus the propagator is

$$\langle \Phi_{ab}(n, p) \Phi_{cd}(-n, -p) \rangle_{S^1 \times R^{d-1}} = \delta_{ad} \delta_{bc} \frac{g_{YM}^2}{\left(\frac{2\pi}{\beta} \left(n + \frac{a-b}{N} \right) \right)^2 + p^2}. \quad (2.4)$$

We have included the effect of the A_0 zero-mode configuration (2.2) in the propagator.⁶

⁴In the case where the spacial manifold is compact rather than R^{d-1} , $\langle |\mathcal{P}|^2 \rangle$ is a more suitable order parameter [38].

⁵We thank S. Minwalla for stressing us that the difference of the configuration of A_0 should be reflected when we probe the bulk geometry by field theory correlators.

⁶The techniques below are reminiscent of those used in the reduced models [58, 59, 60, 61, 62].

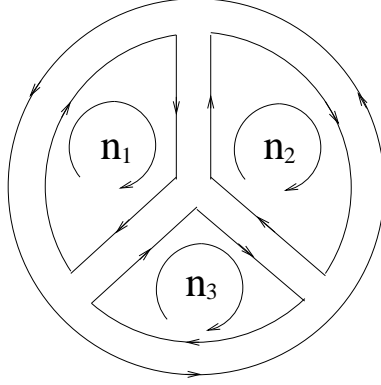


Figure 1: Assigning loop momenta to matrix index loops in 't Hooft's double line representation of gauge theory planar Feynman diagram. In planar diagram the loop momenta are one less than the index loops, so there remains one extra index loop which we chose to be the outer index loop here.

Since we are considering planar limit $N \rightarrow \infty$, we can replace matrix index sums by integrals:

$$\sum_{a_i=1}^N f(a_i) \rightarrow \frac{\beta N}{2\pi} \int_0^{\frac{2\pi}{\beta}} dp_{0i} f(p_{0i}), \quad (2.5)$$

for some function $f(a_i)$. Notice the factor of N in front of the integral. This means that the modification of the propagator (2.4) by the zero-mode of the gauge field A_0 (2.2) does not change the argument of 't Hooft: Each matrix index loop, or face, contributes with a factor of N . Now suppose we calculate correlation functions in this theory. In planar diagrams, we can parameterize the temporal loop momenta by matrix index line notation (see e.g. [60, 63]). For a given planar Feynman diagram with ℓ momentum loops, we have $\ell + 1$ matrix index loop. We assign the loop momentum n_i ($i = 1, \dots, \ell$) to every index loop but one, say $(\ell + 1)$ -th index loop (Fig.1). Correlation functions are calculated by connecting the fields and vertices with the propagators (2.4), and a_i 's only appear in the combination $a_i - a_j$. This doesn't depend on a constant shift to all a_i with some integer q , so one summation of internal indices, which we conventionally choose to be $a_{\ell+1}$, gives just a factor N . Also the index p_{0i} and the loop momentum n_i can always be combined as $p_{0i} + \frac{2\pi n_i}{\beta}$. The origin of this combination is gauge covariance,

the covariant derivative in the Euclidean time direction. Notice that both momentum- and index- flows are associated with a direction (indicated by an arrow) in the 't Hooft's double line representation of Feynman diagrams. Thus we can combine all the other index loop sums, which were replaced by integral in (2.5), with the temporal momentum sums n_i (2.5):

$$\sum_{n_i=-\infty}^{\infty} \int_0^{\frac{2\pi}{\beta}} dp_{0i} f(p_{0i} + \frac{2\pi n_i}{\beta}) = \int_{-\infty}^{\infty} dp_{0i} f(p_{0i}) \quad (2.6)$$

for some function $f(p_{0i})$. Therefore internal loop momentum integrals in calculation of correlation functions on $S^1 \times R^{d-1}$ with the A_0 zero-mode configuration (2.2) become the same as the ones without the S^1 compactification: Suppose we have a M -point function of gauge invariant operators on $R \times R^{d-1}$ as a function of incoming momenta k (we suppress the spatial momenta in the following expressions):⁷

$$\langle \mathcal{O}_1(k_{01}) \cdots \mathcal{O}_M(k_{0M}) \rangle_{R \times R^{d-1}} = G_M(k_{01}, \cdots k_{0M}). \quad (2.7)$$

Then on $S^1 \times R^{d-1}$ we will obtain the same function G_M with incoming momenta $\frac{2\pi m}{\beta}$:

$$\langle \mathcal{O}_1(\frac{2\pi m_1}{\beta}) \cdots \mathcal{O}_M(\frac{2\pi m_M}{\beta}) \rangle_{S^1 \times R^{d-1}} = G_M(\frac{2\pi m_1}{\beta}, \cdots, \frac{2\pi m_M}{\beta}). \quad (2.8)$$

The only difference is that the incoming momenta are discrete. Let the Fourier transform of $G_M(k)$ be $G_M(\tau)$:

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk_0 G_M(k_0) e^{ik_0 \tau} = G_M(\tau) \quad (2.9)$$

where we schematically picked up one incoming momentum, but the calculation is the same for all the incoming momenta. Then by the Poisson resummation formula,

$$\frac{\sqrt{2\pi}}{\beta} \sum_{m=-\infty}^{\infty} G_M(\frac{2\pi m}{\beta}) e^{i\frac{2\pi m}{\beta} \tau} = \sum_{n=-\infty}^{\infty} G_M(\tau + \beta n). \quad (2.10)$$

This means if one reads off some geometry of the bulk from a field theory on $R \times R^{d-1}$, on $S^1 \times R^{d-1}$ with zero Polyakov loop expectation value one just finds the same geometry except that it has a periodic identification in τ direction.

⁷All the correlation functions studied in this article will be connected diagrams and we will not explicitly mention that hereafter. For non-connected diagrams our arguments straightforwardly apply to each connected components.

Since correlation functions on $S^1 \times R^{d-1}$ in momentum space have the same form as in the $R \times R^{d-1}$ case, the gluing procedure of [2] works exactly in the same way in confined phase.

As an example, let us calculate the following simple three point function in free field theory (Fig.2):

$$\langle \text{Tr} \Phi^2(\frac{2\pi m_1}{\beta}, k_1) \text{Tr} \Phi^2(\frac{2\pi m_2}{\beta}, k_2) \text{Tr} \Phi^2(\frac{2\pi m_3}{\beta}, k_3) \rangle_{S^1 \times R^{d-1}} \quad (2.11)$$

under the A_0 zero-mode configuration (2.2).⁸ Up to the total momentum conservation delta function $\delta_{m_1+m_2+m_3,0} \delta(k_1 + k_2 + k_3)$ this is

$$\sum_{a_1=1}^N \sum_{a_2=1}^N \frac{1}{\beta} \sum_{n=-\infty}^{\infty} \int d^{d-1} p \frac{g_{YM}^6}{\left(\left(\frac{2\pi n + \frac{a_1 - a_2}{N}}{\beta} \right)^2 + p^2 \right) \left(\left(\frac{2\pi(n-m_2) + \frac{a_1 - a_2}{N}}{\beta} \right)^2 + (p - k_2)^2 \right) \left(\left(\frac{2\pi(n+m_3) + \frac{a_1 - a_2}{N}}{\beta} \right)^2 + (p + k_3)^2 \right)}. \quad (2.12)$$

Then we turn loop index a_1 summation into integral $\int_0^{\frac{2\pi}{\beta}} dp_0$ and combine the p_0 integral with the temporal momentum n summation. The result is

$$N^2 \cdot \lambda^3 N^{-3} \int dp_0 d^{d-1} p \frac{1}{(p_0^2 + p^2) \left(\left(p_0 - \frac{2\pi m_2}{\beta} \right)^2 + (p - k_2)^2 \right) \left(\left(p_0 + \frac{2\pi m_3}{\beta} \right)^2 + (p + k_3)^2 \right)}. \quad (2.13)$$

The conversion of the summation over n to the integral gave a factor of N , and redundant summation over a_2 gave another factor of N , together with contributions from propagators resulting a factor of $N^2 \cdot N^{-3}$ which is appropriate for sphere with three punctures. (2.13) is the same form as the one in $R \times R^{d-1}$ case, except that the temporal momenta take discrete values. The three point function on $R \times R^{d-1}$ in position space can be written as [2]

$$\langle \text{Tr} \Phi^2(\tau_1, x_1) \text{Tr} \Phi^2(\tau_2, x_2) \text{Tr} \Phi^2(\tau_3, x_3) \rangle_{R \times R^{d-1}}$$

⁸Actually confined phase is not thermodynamically favored in free field theory on $S^1 \times R^{d-1}$. However, once the phase of a system of interest is known to be in confined phase the following calculation does not depend on the details of the spatial manifold or interactions. Therefore please regard this example as an exhibition of the calculational essence applicable for field theories which do have confined phase.

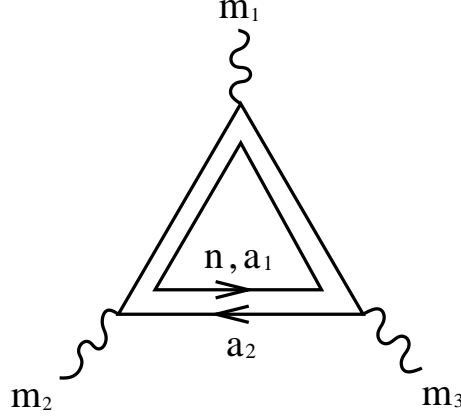


Figure 2:

$$= \lambda^3 N^{-1} \int_0^\infty \frac{dt}{t^{\frac{d}{2}+1}} \int_{-\infty}^\infty d\tau' \int_{-\infty}^\infty d^{d-1}x' \prod_{s=1}^3 K_\Delta(\tau_s, x_s; \tau', x'; t), \quad (2.14)$$

where

$$K_\Delta(\tau, x; \tau', x'; t) = \frac{t^{\frac{\Delta}{2}}}{[t + (\tau - \tau')^2 + (x - x')^2]^\Delta} \quad (2.15)$$

is the usual position space bulk to boundary propagator in AdS_{d+1} for a scalar field corresponding to an operator of dimension Δ , $\Delta = d - 2$ in the case at hand. Then on $S^1 \times R^{d-1}$ we get

$$\begin{aligned} & \langle \text{Tr} \Phi^2(\tau_1, x_1) \text{Tr} \Phi^2(\tau_2, x_2) \text{Tr} \Phi^2(\tau_3, x_3) \rangle_{S^1 \times R^{d-1}} \\ &= \lambda^3 N^{-1} \int_0^\infty \frac{dt}{t^{\frac{d}{2}+1}} \int_{-\infty}^\infty d\tau' \int_{-\infty}^\infty d^{d-1}x' \prod_{s=1}^3 \sum_{n_s=-\infty}^\infty K_\Delta(\tau_s + \beta n_s, x_s; \tau', x'; t) \\ &= \lambda^3 N^{-1} \int_0^\infty \frac{dt}{t^{\frac{d}{2}+1}} \int_0^\beta d\tau' \int_{-\infty}^\infty d^{d-1}x' \prod_{s=1}^3 \sum_{n=-\infty}^\infty \sum_{n_s=-\infty}^\infty K_\Delta(\tau_s + \beta n_s, x_s; \tau' + \beta n, x'; t). \end{aligned} \quad (2.16)$$

In (2.16) we sum over images of the bulk to boundary propagators in AdS_{d-1} periodically identified in τ direction (Fig.3), as we have explained in more general settings. The confined phase is not actually thermodynamically favored in free field theory on $S^1 \times R^d$. However, the above calculation did not depend on the details of the spatial directions. Therefore it can be straightforwardly applied to the case of other spatial manifolds, for

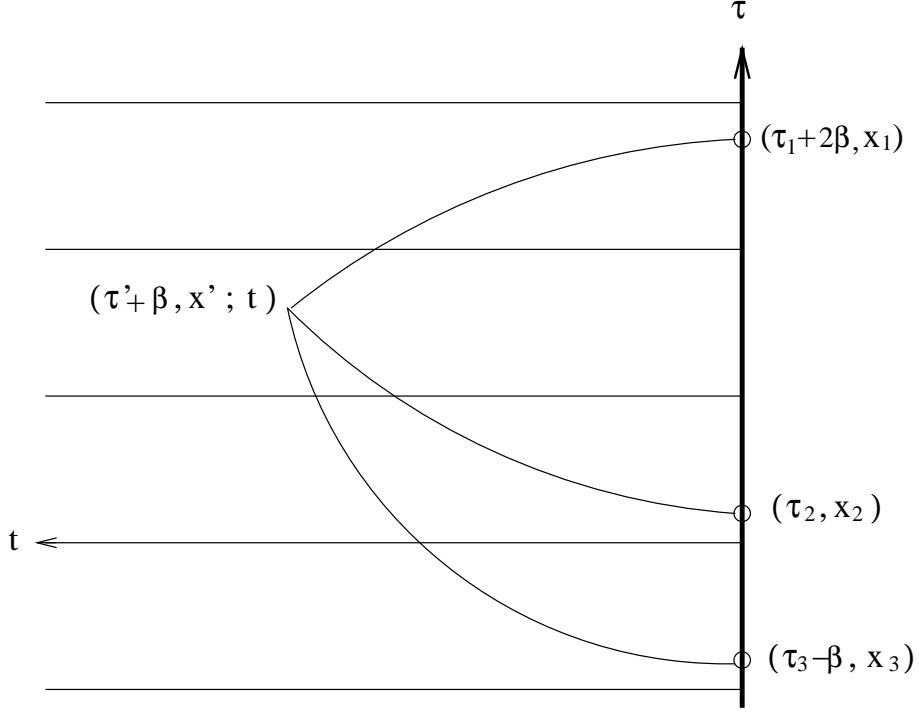


Figure 3: A diagram contributing to (2.16). The field theory correlator can be expressed in terms of the bulk to boundary propagators in AdS_{d+1} periodically identified in τ direction: The bulk to boundary propagators (solid curves) connect images of operators at point $(\tau_1, x_1; 0)$, $(\tau_2, x_2; 0)$, $(\tau_3, x_3; 0)$ and the bulk point $(\tau', x'; t)$.

example to free Yang-Mills theory on S^3 extensively studied in [38, 39]. There the confined phase is realized at low temperature. In this case one just needs to replace p^2 above to Laplacian for conformally coupled scalars on S^3 , and momentum integration to sum over spherical harmonics on S^3 with an appropriate measure factor.

Before closing this section, we emphasize again that a periodic identification in field theory side does not by itself lead to the periodic identification in the dual bulk geometry. The A_0 zero-mode configuration (2.2), which tells that the system under consideration is in confined phase, is crucial for finding the periodically identified bulk geometry.

3 Deconfined Phase vs. Bulk Geometry

Now let us turn to deconfined phase. We will study the case where the zero-mode of A_0 is zero so that the expectation value of the Polyakov loop is one.⁹ Let us take free massless adjoint scalar field Φ on $S^1 \times R^{d-1}$ as an example. For later purpose, let us work in position space. The propagator on $S^1 \times R^{d-1}$ in position space representation can be obtained by summing over images on its covering space $R \times R^{d-1}$:

$$\langle \Phi_{ab}(\tau, x) \Phi_{cd}(0) \rangle_{S^1 \times R^{d-1}} = \sum_{n=-\infty}^{\infty} \langle \Phi_{ab}(\tau + \beta n, x) \Phi_{cd}(0) \rangle_{R \times R^{d-1}}. \quad (3.1)$$

The propagator on $R \times R^{d-1}$ is given by

$$\langle \Phi_{ab}(\tau, x) \Phi_{cd}(0) \rangle_{R \times R^{d-1}} = \delta_{ad} \delta_{bc} \frac{1}{(\tau^2 + x^2)^{\frac{d-2}{2}}}. \quad (3.2)$$

Now let us recall the argument of [2] for reorganizing field theory correlators into closed string amplitudes. The crucial step was the gluing of propagators in each edge of a given Feynman diagram to make up a so-called skeleton graph (Fig.4).

The gluing in position space is seen as follows.¹⁰ Suppose the r -th edge has m_r propagators. Then, it has a contribution proportional to

$$\left(\frac{1}{(\tau^2 + x^2)^{\frac{d-2}{2}}} \right)^{m_r}. \quad (3.3)$$

⁹This may be the most typical configuration, but this is not the most general case.

¹⁰We thank R. Gopakumar for stressing the usefulness of viewing the gluing in position space.

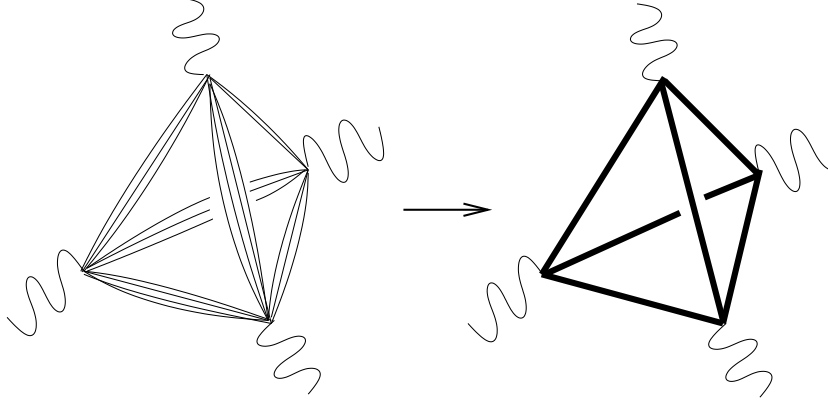


Figure 4: Gluing propagators so that each edge is parameterized by one Schwinger parameter.

One can exponentiate each propagator by a Schwinger parameter to obtain

$$\left(\frac{1}{(\tau^2 + x^2)^{\frac{d-2}{2}}} \right)^{m_r} = \prod_{\mu_r=1}^{m_r} \frac{1}{\Gamma(\frac{d-2}{2})} \int_0^\infty d\sigma_{\mu_r} \sigma_{\mu_r}^{\frac{d-2}{2}-1} e^{-\sigma_{\mu_r}(\tau^2 + x^2)}. \quad (3.4)$$

If we insert the identity

$$1 = \int_0^\infty d\sigma_r \delta\left(\sigma - \sum_{\mu_r=1}^{m_r} \sigma_{\mu_r}\right) \quad (3.5)$$

to (3.4) and then change the variables to $\alpha_{\mu_r} = \frac{\sigma_{\mu_r}}{\sigma_r}$, we obtain

$$\begin{aligned} & \prod_{\mu_r=1}^{m_r} \frac{1}{\Gamma(\frac{d-2}{2})} \int_0^\infty d\sigma_r \sigma_r^{\frac{d-2}{2}m_r} e^{-\sigma_r(\tau^2 + x^2)} \int_0^1 d\alpha_{\mu_r} \delta\left(\sigma \left(1 - \sum_{\mu_r=1}^{m_r} \alpha_{\mu_r}\right)\right) \\ &= \left(\frac{1}{\Gamma(\frac{d-2}{2})} \right)^{m_r} \int_0^\infty d\sigma_r \sigma_r^{\frac{d-2}{2}m_r-1} e^{-\sigma_r(\tau^2 + x^2)} \cdot \prod_{\mu_r=1}^{m_r} \int_0^1 d\alpha_{\mu_r} \delta\left(1 - \sum_{\mu_r=1}^{m_r} \alpha_{\mu_r}\right). \end{aligned} \quad (3.6)$$

The alpha integrals factor out to give an overall constant $\frac{(\Gamma(\frac{d-2}{2}))^{m_r}}{\Gamma(\frac{d-2}{2}m_r)}$ and one effective Schwinger parameter σ_r remains. Thus we have "glued" Schwinger parameters into an effective Schwinger parameter σ_r . By Fourier transforming it into momentum space one gets the expression given in [2] and σ_r is related to the momentum space effective Schwinger parameter τ_r in [2] as $\sigma_r = \frac{1}{\tau_r}$. On the other hand, one could have exponentiated (3.3) at once using just one Schwinger parameter:

$$\left(\frac{1}{(\tau^2 + x^2)^{\frac{d-2}{2}}} \right)^{m_r} = \frac{1}{\Gamma(\frac{d-2}{2}m_r)} \int_0^\infty d\sigma_r \sigma_r^{\frac{d-2}{2}m_r-1} e^{-\sigma_r(\tau^2 + x^2)}. \quad (3.7)$$

This seems to suggest that in position space the multiplication of propagators itself may be identified with the "gluing".

Once propagators in each edge are glued and each edge in the resulting skeleton graph is assigned a Schwinger parameter, the space of the Schwinger parameters gives a cell decomposition of $\mathcal{M}_{g,n} \times R_+^n$ as argued in [2], where $\mathcal{M}_{g,n}$ is a moduli space of n -punctured genus g Riemann surface. The appearance of the moduli space $\mathcal{M}_{g,n}$ is a strong indication that large N field theory amplitudes do reorganize themselves into closed strings. See [1, 2, 3, 4] for more details.

How gluing works in the case of $S^1 \times R^{d-1}$? One may try to follow the arguments of [2] which were done in momentum representation (see also [64]). However, the difference between $R \times R^{d-1}$ and $S^1 \times R^{d-1}$ manifests itself in several stages and one cannot straightforwardly follow the steps in the $R \times R^{d-1}$ case. We let the interested readers to try to follow the arguments of [2] and [64] in the case of $S^1 \times R^{d-1}$ and see where the differences appear (however see the remarks below). Here, instead, we work in position space and try to understand the reason why on $S^1 \times R^{d-1}$ gluing procedure is not straightforward. First, we would like to interpret the propagator (3.2) as a propagation of a "string bit" Φ in AdS space which makes up a closed string. This is plausible since up to the matrix indices the functional dependence of the propagator is fixed by conformal symmetry, and the function of the form in (3.2);

$$\frac{1}{(\tau^2 + x^2)^{\frac{d-2}{2}}} \quad (3.8)$$

can always be understood in terms of bulk to boundary propagator in AdS_{d+1} . Then, if we use the righthand side of (3.1) to calculate correlators, it may be interpreted as summing over contributions of each bit propagating in periodically identified AdS space going around the S^1 direction for a different number of times. In that case, the resulting closed string worldsheet will be wildly torn apart. This is somewhat reminiscent of the interpretation of the Hagedorn transition in string theory [48, 49, 50, 51]. This will make the closed string interpretation in AdS space inappropriate in deconfined phase. We identify this as the reason why on $S^1 \times R^3$ the gluing procedure is not straightforward.

However, now we would like to argue that one can nevertheless "glue" the propagators and give a closed string interpretation, but not in the AdS space but in a *different* bulk geometry. For that we interpret the *result* of the summation in (3.1) as coming from a

bit propagator in this *new* bulk geometry. This interpretation with a new bulk geometry is also reminiscent of the speculation about the phase after the Hagedorn transition [65, 66, 67]. Since there is no summation over images any more, there should not be non-contradictable circle in the new geometry. This conclusion is in good accordance with the criteria of confinement/deconfinement from string worldsheet consideration in the bulk [55]. Then, the gluing procedure in this case will be multiplication of position space propagators in each edge, as was the case in $R \times R^{d-1}$. Geodesic approximation illustrates this interpretation: For large J it gives

$$\begin{aligned} \langle \text{Tr} \Phi^J(p) \text{Tr} \Phi^J(q) \rangle_{S^1 \times R^{d-1}} &\sim \langle \Phi(p) \Phi(q) \rangle_{S^1 \times R^{d-1}}^J \\ &\sim \left. e^{-M D_{reg}(p,q)} \right|_{\text{new geometry}} = \left(e^{-m D_{reg}(p,q)} \right)^J \Big|_{\text{new geometry}} + \mathcal{O}(1/J) \end{aligned} \quad (3.9)$$

where $M = \frac{1}{R} \sqrt{\Delta(\Delta - d)}$, (with $\Delta = J(d-2)/2$) is a mass of the bulk particle corresponding to the operator $\text{Tr} \Phi^J$ according to the *AdS/CFT* dictionary [13, 14]. $m = (d-2)/2R$ is interpreted as a mass of a string bit. $D_{reg}(p, q)$ is a regularized geodesic distance in the new geometry between the points p and q on the boundary. The formula (3.9) tells us that the gluing of string bit geodesics corresponds to taking a single geodesic with the effective mass given by a sum of all the masses of the string bits, which results in multiplying field theory propagators. See Fig.5.

The reason why the gluing didn't work straightforwardly in momentum space was as follows. If one naively tries to follow the argument of [2] in momentum space, it means he/she is using the Fourier transform of (3.2) as a propagator instead of the lefthand side of (3.1). Then the gluing does not work, at least straightforwardly, as we have argued above.

Although we don't know a good parameterization of Schwinger parameters which is convenient for comparing with a conjectural closed string theory amplitudes on this new background geometry,¹¹ one can certainly assign one parameter to each edge, which we also call "Schwinger parameter", once propagators in each edge are glued to make a skeleton graph.

We will examine the above picture by studying a simple example in the next section.

¹¹In the case of field theory on R^d whose dual is supposed to be a closed string theory on AdS_{d+1} , the correspondence between Schwinger parameters and the moduli space of string worldsheet has largely developed in the recent investigation [3].

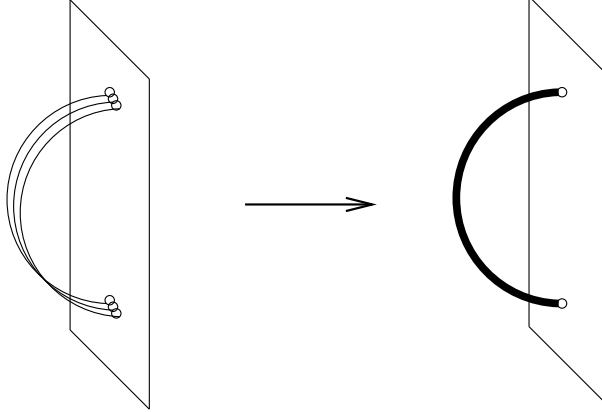


Figure 5: The gluing of a two point function in the "new geometry" in geodesic approximation. The sum of the contributions from the propagation of each string bit can be expressed as contribution from a single geodesic with the effective mass being the sum of the mass of the string bits.

4 CFT on $S^1 \times R$ and the Dual Bulk Geometries

4.1 Free Scalar CFT on $S^1 \times R$

In this section we will study free massless scalar field theory on $S^1 \times R$. But before going into the $S^1 \times R$ case, it is useful to recall the $R \times R$ case, the covering space of $S^1 \times R$. In particular, we will recall how the geometry of AdS space is encoded in conformal field theory correlators.

Let τ be the Euclidean time coordinate and x be the spatial coordinate on R . We define $z = \tau + ix$. The correlation functions are given by¹²

$$\langle \partial\Phi_{ab}(z)\partial\Phi_{cd}(0) \rangle_{R \times R} = \delta_{ad}\delta_{bc} \frac{1}{z^2}, \quad (4.1)$$

$$\langle \bar{\partial}\Phi_{ab}(\bar{z})\bar{\partial}\Phi_{cd}(0) \rangle_{R \times R} = \delta_{ad}\delta_{bc} \frac{1}{\bar{z}^2}. \quad (4.2)$$

¹²In two dimension the scalar field Φ is not a conformal operator so we will consider $\partial\Phi$ ($\bar{\partial}\Phi$) instead.

From the above, we can calculate two point functions like¹³

$$\langle \text{Tr}(\partial\Phi)^J(\bar{\partial}\Phi)^J(z, \bar{z}) \text{Tr}(\partial\Phi)^J(\bar{\partial}\Phi)^J(0) \rangle_{R \times R} \sim \left(\frac{1}{z^2}\right)^J \left(\frac{1}{\bar{z}^2}\right)^J. \quad (4.3)$$

This coincides with the geodesic approximation in AdS_3 space

$$ds^2 = r^2 \left(\frac{d\tau^2 + dx^2}{\beta^2} \right) + R^2 \frac{dr^2}{r^2} \quad (4.4)$$

in large J limit (see Appendix A): The geodesic approximation in this metric gives

$$\langle \text{Tr}(\partial\Phi)^J(\bar{\partial}\Phi)^J(z, \bar{z}) \text{Tr}(\partial\Phi)^J(\bar{\partial}\Phi)^J(0) \rangle_{R \times R} \sim \left(\frac{1}{z^2}\right)^{\sqrt{J(J-1)}} \left(\frac{1}{\bar{z}^2}\right)^{\sqrt{J(J-1)}} + \mathcal{O}(1/J). \quad (4.5)$$

which coincides with (4.3) up to $\mathcal{O}(1/J)$ terms.¹⁴

Now let us study the free massless scalar field theory on $S^1 \times R$, where the Euclidean time τ is compactified on S^1 with period β . By "free", we mean the dimensionless parameter $\beta^2 \lambda \rightarrow 0$, where $\lambda = g_{YM}^2 N$ is the 't Hooft coupling. This system is in deconfined phase and the expectation value of the zero-mode of A_0 is zero. Thus the propagators on $S^1 \times R$ can be obtained by summing over images of its universal covering $R \times R$:

$$\langle \partial\Phi_{ab}(z) \partial\Phi_{cd}(0) \rangle_{S^1 \times R} = \sum_{n=-\infty}^{\infty} \langle \partial\Phi_{ab}(z + \beta n) \partial\Phi_{cd}(0) \rangle_{R \times R} \quad (4.6)$$

$$= \delta_{ad} \delta_{bc} \sum_{n=-\infty}^{\infty} \frac{1}{(z + \beta n)^2} \quad (4.7)$$

$$= \delta_{ad} \delta_{bc} \frac{\pi^2}{\beta^2 \sin^2 \frac{\pi z}{\beta}}, \quad (4.8)$$

$$\langle \bar{\partial}\Phi_{ab}(\bar{z}) \bar{\partial}\Phi_{cd}(0) \rangle_{S^1 \times R} = \sum_{n=-\infty}^{\infty} \langle \bar{\partial}\Phi_{ab}(\bar{z} + \beta n) \bar{\partial}\Phi_{cd}(0) \rangle_{R \times R} \quad (4.9)$$

$$= \delta_{ad} \delta_{bc} \sum_{n=-\infty}^{\infty} \frac{1}{(\bar{z} + \beta n)^2} \quad (4.10)$$

$$= \delta_{ad} \delta_{bc} \frac{\pi^2}{\beta^2 \sin^2 \frac{\pi \bar{z}}{\beta}}. \quad (4.11)$$

¹³For gauge invariance we may better use covariant derivatives $D\Phi$ ($\bar{D}\Phi$). However, we can choose the gauge $\partial_\tau A_0 = 0$, $A_x = 0$, and here we are considering the phase where the zero-mode of A_0 is zero.

¹⁴One may use boundary to bulk propagators instead of the geodesic approximation to refine the match between field theory side and bulk side in this example. Since we are interested in the general picture we can extract rather than a particular nature of this simple geometry, we content ourselves with the geodesic approximation.

Using the above, typically we find two point functions like

$$\langle \text{Tr}(\partial\Phi)^J(\bar{\partial}\Phi)^J(z, \bar{z}) \text{Tr}(\partial\Phi)^J(\bar{\partial}\Phi)^J(0) \rangle \sim \left(\frac{1}{\cosh(\frac{2\pi x}{\beta}) - \cos \frac{2\pi \tau}{\beta}} \right)^{2J}. \quad (4.12)$$

In large J limit this coincides with the result obtained from geodesic approximation (see Appendix A) in Euclidean AdS_3 metric in the static coordinates (with its Euclidian time in AdS sense being the x direction)

$$ds^2 = r^2 \frac{d\tau^2}{\beta^2} + (r^2 + R^2) \frac{dx^2}{\beta^2} + \frac{dr^2}{\frac{r^2}{R^2} + 1}. \quad (4.13)$$

It is important to notice that this geometry is *different* from the geometry (4.4) periodically identified in τ direction. Although both are locally AdS , the choice of the Euclidean time direction according to which the energy is defined is different in both cases. Also, the geometry (4.4) with the periodic identification has a non-contractible circle in τ direction whereas (4.13) does not. (4.13) is an example of what we called "new geometry" in the previous section. In this simple example the new geometry is again locally AdS_3 , but it will be different for different cases, like free fields in higher dimension etc. In the next subsection we will argue that this new geometry is thermodynamically favored at finite temperature in the saddle point approximation of Euclidean path integral gravity, in accordance with the discussions in the previous section.

4.2 Zero Temperature Phase Transition in the Bulk

The boundary geometry $S^1 \times R$ may admit two bulk saddle points, depending on the gravitational action I which is a functional of the bulk metric. We will comment on the action I shortly. One saddle point will be the Euclidean AdS_3 in the Poincare coordinates (4.4) with periodic identification in τ (thermal AdS):

$$ds^2 = r^2 \left(\frac{d\tau^2 + dx^2}{\beta^2} \right) + R^2 \frac{dr^2}{r^2}. \quad (4.14)$$

The other is (4.13), the Euclidean AdS_3 in the static coordinates (with its Euclidian time in AdS sense being the x direction):

$$ds^2 = r^2 \frac{d\tau^2}{\beta^2} + (r^2 + R^2) \frac{dx^2}{\beta^2} + \frac{dr^2}{\frac{r^2}{R^2} + 1}. \quad (4.15)$$

We can make a change of coordinate $r^2 \rightarrow r^2 - R^2$ in (4.15) to obtain the following form

$$ds^2 = (r^2 - R^2) \frac{d\tau^2}{\beta^2} + r^2 \frac{dx^2}{\beta^2} + \frac{dr^2}{\frac{r^2}{R^2} - 1}. \quad (4.16)$$

The metric (4.16) is a special case of the one studied in [55], which simply reduces to AdS_3 in the case of three bulk dimension. We will call (4.14) geometry I and (4.15) or (4.16) geometry II. Note that the geometry II would have a conical singularity unless the period of τ is β . The geometry I has a non-contractible circle along τ direction (see Fig.6). There is no surface with a disk topology in the bulk that ends on a boundary thermal circle, and from the dictionary of AdS/CFT correspondence this corresponding to the zero Polyakov loop expectation value in the CFT side [55]. The geometry II covers whole AdS_3 and there's no non-contractible circle (see Fig.7), corresponding to the non-zero Polyakov loop expectation value in the CFT side.

In order to discuss thermodynamics in the Euclidean path integral formulation of gravity, we need to know the action I . We expect this action to be ultimately derived from a conjectural closed string theory dual to the free field theory. The closed string loop correction is suppressed by $\frac{1}{N}$ in the planar limit, but the string α' correction will be large since the dual geometry has a curvature scale around the order of the string scale. Although we may expect AdS_3 to be an exact string background, we do not know how the string correction to the action would be. However, for a constant curvature space (locally AdS_3) the classical gravitational action is proportional to the volume if it is generally covariant, i.e. made out of generally covariant combinations of metric and curvature tensor and its covariant derivatives. Schematically,

$$I \propto \int dx^3 \sqrt{g} \quad (\text{for constant curvature spaces}). \quad (4.17)$$

We assume the coefficient for the proportionality is positive. We also assume that the geometry I and II are the only two minima of this action. Actually, the above volume (4.17) is infinite so we need to regularize it. In order for that we introduce a cut off r_{reg} in the radial coordinate r and take the difference of the two volumes $V_{II}(r_{reg})$ and $V_I(r_{reg})$ corresponding to two geometries II and I respectively [68, 55]:

$$V_{II}(r_{reg}) - V_I(r_{reg}) = \frac{R}{\beta^2} \int_0^\beta d\tau \int_R^{r_{reg}} dr \int_0^L dx \, r - \frac{R}{\beta^2} \int_0^{\beta'} d\tau \int_0^{r_{reg}} dr \int_0^L dx \, r. \quad (4.18)$$

We require the physical circumference of the time direction to be equal at $r = r_{reg}$

$$\beta\sqrt{r_{reg}^2 - R^2} = \beta' r_{reg}. \quad (4.19)$$

Then the difference of the volumes per unit length in x direction is given by

$$\lim_{r_{reg} \rightarrow \infty} \frac{V_{II}(r_{reg}) - V_I(r_{reg})}{L} = -\frac{R^3}{4\beta}. \quad (4.20)$$

The above means we use the geometry I as a reference point to measure free energy. This may be natural because this is the minimum free energy solution at zero-temperature. (4.20) means the action of the geometry II has negative free energy, so in the saddle point approximation as soon as we put the system in finite temperature the geometry II is preferred. This is in good accordance with the thermodynamics of field theory side and the discussions in section 3.¹⁵

The gluing, which was crucial for the closed string picture, is possible only in the "new" geometry II which corresponds to deconfined phase. Notice that in our approach one can see that the instability of the geometry I is a direct consequence of the instability of the symmetric configuration of the zero-mode of the temporal gauge field A_0 (2.2): Once the symmetric A_0 configuration ceases to be stable, field theory correlators start seeing a different geometry.

The energy density E per unit length of the geometry II is given by

$$E = \frac{\partial}{\partial \beta} I \propto \frac{R^3}{4\beta^2} = \frac{R^3}{4} T^2 \quad (4.21)$$

and entropy density S is

$$S = \beta E - I \propto \frac{R^3}{2\beta} = \frac{R^3}{2} T. \quad (4.22)$$

Thus up to the coefficients the geometry II reproduces the results expected for free field theory in two dimension. So the assumptions on the action (4.17) qualitatively reproduce the field theory results.

¹⁵Precisely speaking, the reason it appears as a zero-temperature transition is that we took the high temperature limit $\beta^2 \lambda \rightarrow 0$. The structure of the phase transition is actually hidden at $\beta^2 \lambda \sim 1$ where our free field description is not valid. As long as we are interested in the high energy phase $\beta^2 \lambda \ll 1$ we can use the free field description.

If we rescale the coordinate $r \rightarrow \frac{\beta}{R}r$, the metric (4.14) becomes

$$ds^2 = \frac{r^2}{R^2}(d\tau^2 + dx^2) + R^2 \frac{dr^2}{r^2} \quad (4.23)$$

whereas the metric (4.16) becomes

$$ds^2 = \left(\frac{r^2}{R^2} - \frac{R^2}{\beta^2} \right) d\tau^2 + \frac{r^2}{R^2} dx^2 + \frac{dr^2}{\frac{r^2}{R^2} - \frac{R^2}{\beta^2}}. \quad (4.24)$$

In this form it is clear that in the low temperature limit ($\beta \rightarrow \infty$) the metric (4.24) reduces to the metric (4.23).

4.3 Deconfinement, Hagedorn Transition, String Bits and Gluing

Now let us examine the discussions of section 3 in this example. We interpret each field Φ as a string bit that makes up a closed string. Then the sum (4.7) ((4.10)) means in geometry I correlation functions are obtained by summing over contributions of diagrams where each bit winds the thermal circle for different times (Fig.6). When each bit wind the thermal circle for different times, string worldsheet interpretation may not be appropriate. However, we can interpret (4.8) ((4.11)) as coming from a single propagator of bit on a different geometry II. In geometry II there is no non-contractible circle and hence there is no summation over the winding modes (Fig.7). Then those bits will be able to make up a closed string worldsheet. We interpret multiplying field theory propagators (4.8) ((4.11)) as "gluing" of string bits into a closed string (Fig.8).

The above picture is very much like the interpretation of the Hagedorn transition in string theory. Indeed, when the spatial manifold is some other manifold, for example S^3 , the field theory do have a Hagedorn density of states and this phase transition is at some non-zero temperature [15, 18, 38, 39].¹⁶ In this case the lightest mass of the fields after Kaluza-Klein compactification of the spatial manifold is most relevant for determining the Hagedorn temperature. If we regard spatial line R in our model as an infinite interval limit of a segment with Dirichlet boundary conditions on both ends, the Hagedorn temperature is proportional to the inverse of the length of the segment and

¹⁶The large N limit makes the phase transition possible [69, 70] on the compact space S^3 .

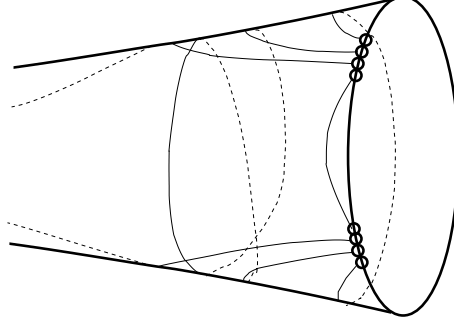


Figure 6: Schematic figure of a contribution to $\langle \text{Tr}(\partial\Phi)^2(\bar{\partial}\Phi)^2 \text{Tr}(\partial\Phi)^2(\bar{\partial}\Phi)^2 \rangle_{S^1 \times R}$ coming from "string bits" propagating in the geometry I. The x -direction is suppressed in the figure. Small circles express "string bits" $\partial\Phi$ ($\bar{\partial}\Phi$) and lines connecting the small circles are paths of the string bits. When each bit is winding τ direction for different times, the string will be wildly torn apart and closed string picture may not be adequate.

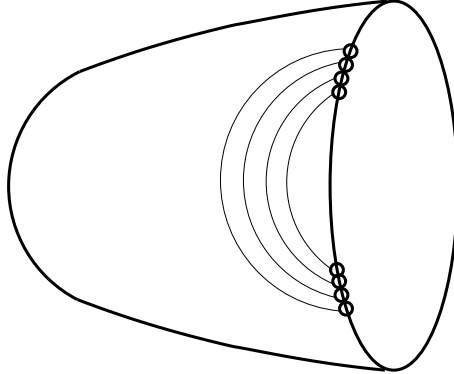


Figure 7: The string bits propagating in the geometry II. In geometry II there is no non-contractible loop.

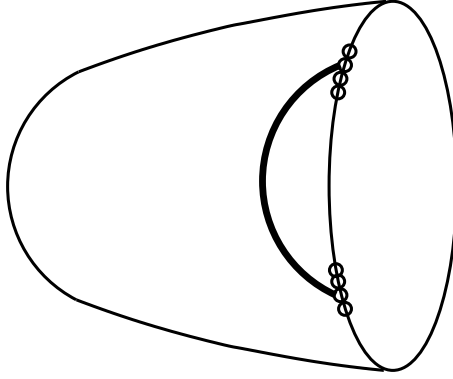


Figure 8: One can "glue" string bits into a closed string in the geometry II.

we may regard the zero-temperature phase transition as a limit of the finite Hagedorn temperature.¹⁷

5 Summary and Future Directions

In this article we analyzed the reorganization of field theory correlators to closed string amplitudes in Euclidean thermal field theory and studied how the dual bulk geometry is encoded on them. The expectation value of the Polyakov loop which is an order parameter of confinement-deconfinement phase is directly encoded on the dual bulk geometry seen by the field theory correlators. Once the Polyakov expectation value is correctly taken into account, the gluing, which was a key step for the reorganization of field theory correlators to closed string amplitudes in [2, 3, 4], is straightforward in confined phase. In deconfined phase the gluing was not straightforward in momentum space. We reexamined the meaning of the gluing and argued that the gluing is still possible if one correctly chooses the dual bulk geometry. We presented free massless scalar field theory on $S^1 \times R$ as a concrete realization of our arguments. In our approach, the instability of a configuration of the zero-mode of the temporal gauge field A_0 at the point of phase transition is directly *translated* into the instability of the bulk geometry.

¹⁷In higher dimension, $S^1 \times R^{d-1}$ ($d \geq 3$) can be regarded as a limit of $S^1 \times S^{d-1}$ where the radius of S^{d-1} goes to infinity. Then the zero-temperature phase transition in $S^1 \times R^{d-1}$ can be regarded as a limit of the Hagedorn transition in $S^1 \times S^{d-1}$.

Studying other compactifications with different field contents is of course interesting. Four dimensional Yang-Mills theory on R^3 is important for its direct relevance to the real world. Compactification to S^3 is also interesting. Even at the weak coupling the phase structure of it qualitatively resembles that of gravity, and it may be continuously continued to them in the strong coupling [38, 39] (see also [71]). The dual geometry of confined phase at low temperature is thermal AdS_5 , as we have discussed in general context. Finding the geometry which corresponds to deconfined phase in this case at weak coupling will not be so easy. However, in the free field limit it may still be possible to find the geometry dual to the deconfined phase, as suggested from the tractability of the field theory side in this limit.

Our analysis was in the zero 't Hooft coupling limit, and in this limit the string bits, or Φ fields, are just loosely tied together by Gauss' law constraint. This corresponds to a tensionless limit of the dual closed string theory. Turning finite 't Hooft coupling will makes bits bind together and their configuration will be more string like. This corresponds to introducing a finite tension in the dual closed string. In flat space the Hagedorn temperature is governed by string tension, and it should be also relevant in the asymptotically AdS spaces. In particular, the Hagedorn transition and the phase transition in Euclidean gravity (Hawking-Page transition [68]) are observed to be separated after including the finite coupling effect [38, 39]. It seems in all known cases at finite coupling the Hawking-Page transition always occurs before the Hagedorn transition, but there is no general argument for it must be so.¹⁸ The indication of this fact to the similarity between the behavior of the string bits and the interpretation of the Hagedorn transition in string theory is not clear to us yet. But the identification of difficulty of the gluing with the behavior of the string bits winding the thermal circle for different times applies no matter whether it is related to the Hagedorn transition or not. The Hagedorn transition interpretation may still apply through the interpretation of the Hawking-Page transition as a local Hagedorn transition, as proposed in [65, 66, 67]. It is very important to clarify those issues at finite coupling.

It is also interesting to study other modifications in the field theory side and look for the corresponding dual geometries. For example, massive deformation of field theory

¹⁸According to S. Minwalla in his lecture at the Strings Meeting 2004 at Khajuraho.

meets obstruction for the gluing procedure similar to the one we met in deconfined phase if one works in momentum space, and one is urged to find an appropriate deformation in the dual geometry. Studying massive case will be also useful for applying our method to the two-dimensional matrix models. In particular, it will be interesting to study the matrix model for two dimensional black holes [72], where the Polyakov loop also plays key roles [73]. Also, recently in the context of perturbative string theory in the orbifold of flat space the authors of [74] identified the mechanism of chronology protection with Hagedorn-like transition before closed null curves form. On the other hand, chronology protection had also been studied through holographic dual descriptions by boundary field theory (see the references in [74]). It will be interesting to relate those two approaches through our method.

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Appendix

A Geodesics in AdS_{d+1}

Here we give some formula for (regularized) geodesic distance in AdS_{d+1} used in section 4. The method is fairly standard, see e.g. [75, 76].

(Euclidean) AdS_{d+1} can be described as a surface in $d+2$ dimensional flat space with signature $(-, +, \dots, +)$:

$$\eta_{AB}y^Ay^B = -R^2, \quad \eta_{AB} = \text{diag}(-, +, \dots, +). \quad (\text{A.1})$$

The geodesic distance $D(p, q)$ between points p and q with coordinates y_p and y_q respectively is given by

$$D(p, q) = R \cosh^{-1} \frac{\langle y_p, y_q \rangle}{R^2} \quad (\text{A.2})$$

where $\langle y_p, y_q \rangle = -\eta_{AB}y_p^Ay_q^B$.

Let us calculate the geodesic distance of two points which are on the boundary of the Poincare coordinates. The coordinatization of Poincare coordinates is given by

$$\begin{aligned} y^0 &= \frac{1}{2u} \left(1 + \frac{u^2}{2} (R^2 + \sum_{i=1}^d (x^i)^2) \right), \\ y^d &= \frac{1}{2u} \left(-1 - \frac{u^2}{2} (R^2 - \sum_{i=1}^d (x^i)^2) \right), \\ y^i &= Rux^i \quad (i = 1, \dots, d). \end{aligned} \quad (\text{A.3})$$

Using (A.2) the geodesic distance between two points $p = (u, x)$, $q = (u, 0)$ at large u turns out to be

$$D(p, q) \sim R \log \left[\frac{R^2 u^4}{16} x^2 \right]. \quad (\text{A.4})$$

Thus after subtracting x independent divergent piece, the geodesic approximation gives

$$e^{-MD(p,q)} = \left(\frac{1}{x^2} \right)^{MR} \quad (\text{A.5})$$

where from the AdS/CFT dictionary the mass M is given by [13, 14]

$$M^2 R^2 = \Delta(\Delta - d). \quad (\text{A.6})$$

Δ is the dimension of corresponding operator.

Next let us study the static coordinates. It is obtained by the following coordinate transformation

$$y^0 = \sqrt{r^2 + R^2} \cosh \frac{t}{R}, \quad y^1 = \sqrt{r^2 + R^2} \sinh \frac{t}{R}, \quad y^j = r \Omega_{d-1}^{j-2} \quad (\text{A.7})$$

where Ω_{d-1}^j are angular coordinates on S^{d-1} , we obtain the metric

$$ds^2 = \left(\frac{r^2}{R^2} + 1 \right) dt^2 + \left(\frac{r^2}{R^2} + 1 \right)^{-1} dr^2 + r^2 d\Omega_{d-1}^2. \quad (\text{A.8})$$

The $r \rightarrow \infty$ boundary is S^3 . The leading term of geodesic distance $D(p, q)$ between two points p and q in large r is

$$D(p, q) \sim R \log \frac{r^2}{R^2} \left(\cosh \frac{t_1 - t_2}{R} - \cos(\theta_1 - \theta_2) \right). \quad (\text{A.9})$$

Thus by the saddle point (geodesic) approximation we get

$$e^{-MD(p, q)} \sim \left(\frac{r^2}{R^2 \cosh \frac{t_1 - t_2}{R} - \cos(\theta_1 - \theta_2)} \right)^{MR}. \quad (\text{A.10})$$

By taking $r \rightarrow \infty$ limit rescaling the divergent piece $\frac{r^2}{R^2}$, we arrive at (4.12) by identifying $\frac{t}{R} = \frac{x}{\beta}$, $\theta = \frac{\tau}{\beta}$, in the large J limit $MR = \sqrt{J(J-1)} \sim J + \mathcal{O}(1/J)$ for $d = 2$.

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